

EXOTIC DEFORMATION QUANTIZATION

VALENTIN OVSIENKO

1 Introduction

Let \mathcal{A} be one of the following commutative associative algebras: the algebra of all smooth functions on the plane: $\mathcal{A} = C^\infty(\mathbf{R}^2)$, or the algebra of polynomials $\mathcal{A} = \mathbf{C}[p, q]$ over \mathbf{R} or \mathbf{C} . There exists a non-trivial formal *associative* deformation of \mathcal{A} called the *Moyal \star -product* (or the standard \star -product). It is defined as an associative operation $\mathcal{A}^{\otimes 2} \rightarrow \mathcal{A}[[\hbar]]$ where \hbar is a formal variable. The explicit formula is:

$$F \star_{\hbar} G = FG + \sum_{k \geq 1} \frac{(i\hbar)^k}{2^k k!} \{F, G\}_k, \quad (1)$$

where $\{F, G\}_1 = \frac{\partial F}{\partial p} \frac{\partial G}{\partial q} - \frac{\partial F}{\partial q} \frac{\partial G}{\partial p}$ is the standard Poisson bracket, the higher order terms are:

$$\{F, G\}_k = \sum_{i=0}^k (-1)^i \binom{k}{i} \frac{\partial^k F}{\partial p^{k-i} \partial q^i} \frac{\partial^k G}{\partial p^i \partial q^{k-i}}. \quad (2)$$

The Moyal product is the *unique* (modulo equivalence) non-trivial formal deformation of the associative algebra \mathcal{A} (see [13]).

Definition 1. A formal associative deformation of \mathcal{A} given by the formula (1) is called a \star -product if:

- 1) the first order term coincides with the Poisson bracket: $\{F, G\}_1 = \{F, G\}$;
- 2) the higher order terms $\{F, G\}_k$ are given by differential operators vanishing on constants: $\{1, G\}_k = \{F, 1\}_k = 0$;
- 3) $\{F, G\}_k = (-1)^k \{G, F\}_k$.

Definition 2. Two \star -products \star_{\hbar} and \star'_{\hbar} on \mathcal{A} are called *equivalent* if there exists a linear mapping $A_{\hbar} : \mathcal{A} \rightarrow \mathcal{A}[[\hbar]]$:

$$A_{\hbar}(F) = F + \sum_{k=1}^{\infty} A_k(F) \hbar^k$$

intertwining the operations \star_{\hbar} and \star'_{\hbar} : $A_{\hbar}(F) \star'_{\hbar} A_{\hbar}(G) = A_{\hbar}(F \star_{\hbar} G)$.

Consider now \mathcal{A} as a Lie algebra: the commutator is given by the Poisson bracket. The Lie algebra \mathcal{A} has a unique (modulo equivalence) non-trivial formal deformation called the *Moyal bracket* or the Moyal \star -commutator: $\{F, G\}_t = \frac{1}{i\hbar}(F \star_{\hbar} G - G \star_{\hbar} F)$, where $t = -\hbar^2/2$.

The well-known De Wilde–Lecomte theorem [4] states the existence of a non-trivial \star -product for an arbitrary symplectic manifold. The theory of \star -products is a subject of *deformation quantization*. The geometrical proof of the existence theorem was given by B. Fedosov [9] (see [8] and [20] for clear explanation and survey of recent progress).

The main idea of this paper is to consider the algebra $\mathcal{F}(M)$ of functions (with singularities) on the cotangent bundle T^*M which are *Laurent polynomials* on the fibers. In contrast to above algebra \mathcal{A} it turns out that for such algebras the standard \star -product is no more unique at least if M is one-dimensional: $\dim M = 1$.

We consider $M = S^1, \mathbf{R}$ in the real case, and $M = \mathcal{H}$ (the upper half-plane) in the holomorphic case. The main result of this paper is an explicit construction of a new \star -product on the algebra $\mathcal{F}(M)$ non-equivalent to the standard Moyal product. This \star -product is equivariant with respect to the Möbius transformations. The construction is based on the bilinear SL_2 -equivariant operations on tensor-densities on M , known as *Gordan transvectants* and *Rankin–Cohen brackets*.

We study the relations between the new \star -product and extensions of the Lie algebra $\text{Vect}(S^1)$.

The results of this paper are closely related to those of the recent work of P. Cohen, Yu. Manin and D. Zagier [3] where a one-parameter family of associative products on the space of classical modular forms is constructed using the same SL_2 -equivariant bilinear operations.

2 Definition of the exotic \star -product

2.1 Algebras of Laurent polynomials. Let \mathcal{F} be one of the following

associative algebras of functions:

$$\mathcal{F} = C[p, 1/p] \otimes C^\infty(\mathbf{R}) \quad \text{or} \quad \mathcal{F} = C[p, 1/p] \otimes \text{Hol}(\mathcal{H})$$

This means, it consists of functions of the type:

$$F(p, q) = \sum_{i=-N}^N p^i f_i(q) \quad (3)$$

where $f_i(q) \in C^\infty(\mathbf{R})$ in the real case, or $f_i(q)$ are holomorphic functions on the upper half-plane \mathcal{H} or $f_i \in C[q]$ (respectively).

We will also consider the algebra of polynomials: $\mathbf{C}[p, 1/p, q]$ (Laurent polynomials in p) .

2.2 Transvectants. Consider the following bilinear operators on functions of one variable:

$$J_k^{m,n}(f, g) = \sum_{i+j=k} (-1)^i \binom{k}{i} \frac{(2m-i)!(2n-j)!}{(2m-k)!(2n-k)!} f^{(i)} g^{(j)} \quad (4)$$

where $f = f(z), g = g(z), f^{(i)}(z) = \frac{d^i f(z)}{dz^i}$.

These operators satisfy a remarkable property: they are equivariant under Möbius (linear-fractional) transformations. Namely, suppose that the transformation $z \mapsto \frac{az+b}{cz+d}$ (with $ad-bc=1$) acts on the arguments as follows:

$$f(z) \mapsto f\left(\frac{az+b}{cz+d}\right)(cz+d)^m, \quad g(z) \mapsto g\left(\frac{az+b}{cz+d}\right)(cz+d)^n,$$

then $J_k^{m,n}(f, g)$ transforms as: $J_k^{m,n}(f, g)(z) \mapsto J_k^{m,n}(f, g)\left(\frac{az+b}{cz+d}\right)(cz+d)^{m+n-k}$.

In other words, the operations (4) are bilinear SL_2 -equivariant mappings on tensor-densities:

$$J_k^{mn} : \mathcal{F}_m \otimes \mathcal{F}_n \rightarrow \mathcal{F}_{m+n-k}$$

where \mathcal{F}_l is the space of tensor-densities of degree $-l$: $\phi = \phi(z)(dz)^{-l}$.

The operations (4) were discovered more than one hundred years ago by Gordan [11] who called them the *transvectants*. They have been rediscovered many times: in the theory of modular functions by Rankin [18] and by Cohen [2] (so-called Rankin-Cohen brackets), in differential projective geometry by Janson and Peetre [12]. The “multi-dimensional transvectants” were defined in [14] in the context of the the Virasoro algebra and symplectic and contact geometry.

2.3 Main definition. Define the following bilinear mapping $\mathcal{F}^{\otimes 2} \rightarrow \mathcal{F}[[\hbar]]$, for $F = p^m f(q), G = p^n g(q)$, where $m, n \in \mathbf{Z}$, put:

$$F \tilde{\star}_{\hbar} G = \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{2^{2k}} p^{(m+n-k)} J_k^{m,n}(f, g) \quad (5)$$

Note, that the first order term coincides with the Poisson bracket.

This operation will be the main subject of this paper. We call it the *exotic \star -product*.

2.4 Remark. Another one-parameter family of operations on modular forms: $f \star^{\kappa} g = \sum_{n=0}^{\infty} t_n^{\kappa}(k, l) J_n^{kl}(f, g)$, where $t_n^{\kappa}(k, l)$ are very interesting and complicated coefficients, is defined in [3].

3 Main theorems

We formulate here the main results of this paper. All the proofs will be given in Sections 4-7.

3.1 Non-equivalence. The Moyal \star -product (1) defines a non-trivial formal deformation of \mathcal{F} . We will show that the formula (5) defines a \star -product non-equivalent to the standard Moyal product.

Theorem 1. *The operation (5) is associative, it defines a formal deformation of the algebra \mathcal{F} which is not equivalent to the Moyal product.*

The associativity of the product (5) is a trivial corollary of Proposition 1 below. To prove the non-equivalence, we will use the relations with extensions of the Lie algebra of vector fields on S^1 : $\text{Vect}(S^1) \subset \mathcal{F}$ (cf. Sec.5).

It is interesting to note, that the constructed \star -product is equivalent to the standard Moyal product if we consider it on the algebra $C^{\infty}(T^*M \setminus M)$ of all smooth functions (not only Laurent polynomials on fibers), cf. Corollary 1 below.

3.2 sl_2 -equivariance. The Lie algebra $sl_2(\mathbf{R})$ has two natural embeddings into the Poisson Lie algebra on \mathbf{R}^2 : the *symplectic Lie algebra* $sp_2(\mathbf{R}) \cong sl_2(\mathbf{R})$ generated by quadratic polynomials (p^2, pq, q^2) and another one with generators: (p, pq, pq^2) which is called the *Möbius algebra*.

It is well-known that the Moyal product (1) is the unique non-trivial formal deformation of the associative algebra of functions on \mathbf{R}^2 equivariant under the action of the symplectic algebra. This means, (1) satisfies the Leibnitz property:

$$\{F, G \star_{\hbar} H\} = \{F, G\} \star_{\hbar} H + G \star_{\hbar} \{F, H\} \quad (6)$$

where F is a quadratic polynomial (note that $\{F, G\}_t = \{F, G\}$ if F is a quadratic polynomial).

Theorem 2. *The product (5) is the unique formal deformation of the associative algebra \mathcal{F} equivariant under the action of the Möbius algebra.*

The product (5) is the unique non-trivial formal deformation of \mathcal{F} satisfying (6) for F from the Möbius sl_2 algebra.

3.3 Symplectomorphism Φ . The relation between the Moyal product and the product (5) is as follows. Consider the symplectic mapping

$$\Phi(p, q) = \left(\frac{p^2}{2}, \frac{q}{p} \right). \quad (7)$$

defined on $\mathbf{R}^2 \setminus \mathbf{R}$ in the real case and on \mathcal{H} in the complex case.

Proposition 1. *The product (5) is the Φ -conjugation of the Moyal product:*

$$F \tilde{\star}_\hbar G = F \star_\hbar^\Phi G := (F \circ \Phi \star_\hbar G \circ \Phi) \circ \Phi^{-1} \quad (8)$$

Remark. The mapping (7) (in the complex case) can be interpreted as follows. It transforms the space of *holomorphic tensor-densities* of degree $-k$ on \mathbf{CP}^1 to the space $C^k[p, q]$ of polynomials of degree k . Indeed, there exists a natural isomorphism $z^n(dz)^{-m} \mapsto p^m q^n$ (where $m \geq 2n$) and $(p^m q^n) \circ \Phi = p^{2m-n} q^n$.

3.4 Operator formalism. The Moyal product is related to the following Weil quantization procedure. Define the following differential operators:

$$\begin{aligned} \hat{p} &= i\hbar \frac{\partial}{\partial q}, \\ \hat{q} &= q \end{aligned} \quad (9)$$

satisfying the canonical relation: $[\hat{p}, \hat{q}] = i\hbar \mathbf{I}$. Associate to each polynomial $F = F(p, q)$ the differential operator $\hat{F} = \text{Sym} F(\hat{p}, \hat{q})$ symmetric in \hat{p} and \hat{q} . The Moyal product on the algebra of polynomials coincides with the product of differential operators: $F \star_\hbar G = \hat{F} \hat{G}$.

We will show that the \star -product (5) leads to the operators:

$$\begin{aligned} \hat{p}^\Phi &= \left(\frac{i\hbar}{2} \right)^2 \Delta, \\ \hat{q}^\Phi &= \frac{1}{4i\hbar} (\Delta^{-1} \circ A + A \circ \Delta^{-1}) \end{aligned} \quad (10)$$

(where $\Delta = \frac{\partial^2}{\partial q^2}$ and $A = 2q \frac{\partial}{\partial q} + 1$ is the dilation operator) also satisfying the canonical relation.

Remark that \hat{p}^Φ and \hat{q}^Φ given by (10) on the Hilbert space $L_2(\mathbf{R})$, are not equivalent to the operators (9) since \hat{q}^Φ is symmetric but not self-adjoint (see [6] on this subject).

3.5 “Symplectomorphic” deformations. Let us consider the general situation.

Proposition 2. *Given a symplectic manifold V endowed with a \star -product \star_\hbar and a symplectomorphism Ψ of V , if there exists a hamiltonian isotopy of Ψ to the identity, then the Ψ -conjugate product \star_\hbar^Ψ defined according to the formula (8) is equivalent to \star_\hbar .*

Corollary 1. *The \star -product (5) considered on the algebra of all smooth functions $C^\infty(T^*\mathbf{R} \setminus \mathbf{R})$ is equivalent to the Moyal product.*

4 Möbius-invariance

In this section we prove Theorem 2. We show that the operations of transvectant (4) are Φ -conjugate of the terms of the Moyal product.

4.1 Lie algebra $\text{Vect}(\mathbf{R})$ and modules of tensor-densities. Let $\text{Vect}(\mathbf{R})$ be the Lie algebra of smooth (or polynomial) vector fields on \mathbf{R} :

$$X = X(x) \frac{d}{dx}$$

with the commutator

$$[X(x) \frac{d}{dx}, Y(x) \frac{d}{dx}] = (X(x)Y'(x) - X'(x)Y(x)) \frac{d}{dx}.$$

The natural embedding of the Lie algebra $sl_2 \subset \text{Vect}(\mathbf{R})$ is generated by the vector fields $d/dx, xd/dx, x^2d/dx$.

Define a 1-parameter family of $\text{Vect}(\mathbf{R})$ -actions on $C^\infty(\mathbf{R})$ given by

$$L_X^{(\lambda)} f = X(x)f'(x) - \lambda X'(x)f(x) \quad (11)$$

where $\lambda \in \mathbf{R}$. Geometrically, $L_X^{(\lambda)}$ is the operator of Lie derivative on *tensor-densities* of degree $-\lambda$:

$$f = f(x)(dx)^{-\lambda}.$$

Denote \mathcal{F}_λ the $\text{Vect}(\mathbf{R})$ -module structure on $C^\infty(\mathbf{R})$ given by (11).

4.2 Transvectant as a bilinear sl_2 -equivariant operator. The operations (4) can be defined as bilinear mappings on $C^\infty(\mathbf{R})$ which are sl_2 -equivariant:

Statement 4.1. For each $k = 0, 1, 2, \dots$ there exists a unique (up to a constant) bilinear sl_2 -equivariant mapping

$$\mathcal{F}_\mu \otimes \mathcal{F}_\nu \rightarrow \mathcal{F}_{\mu+\nu-k}.$$

It is given by $f \otimes g \mapsto J_k^{\mu,\nu}(f, g)$.

Proof: straightforward (cf. [11], [12]).

4.3 Algebra \mathcal{F} as a module over $\text{Vect}(\mathbf{R})$. The Lie algebra $\text{Vect}(\mathbf{R})$ can be considered as a Lie subalgebra of \mathcal{F} . The embedding $\text{Vect}(\mathbf{R}) \subset \mathcal{F}$ is given by:

$$X(x) \frac{d}{dx} \mapsto pX(q).$$

The algebra \mathcal{F} is therefore, a $\text{Vect}(\mathbf{R})$ -module.

Lemma 4.2. The algebra \mathcal{F} is decomposed to a direct sum of $\text{Vect}(\mathbf{R})$ -modules:

$$\mathcal{F} = \oplus_{m \in \mathbf{Z}} \mathcal{F}_m.$$

Proof. Consider the subspace of \mathcal{F} consisting of functions homogeneous of degree m in p : $F = p^m f(q)$. This subspace is a $\text{Vect}(\mathbf{R})$ -module isomorphic to \mathcal{F}_m . Indeed, $\{pX(q), p^m f(q)\} = p^m (Xf' - mX'f) = p^m L_X^{(m)} f$.

4.4 Projective property of the diffeomorphism Φ . The transvectants (4) coincide with the Φ -conjugate operators (2) from the Moyal product:

Proposition 4.3. Let $F = p^m f(q), G = p^n g(q)$, then

$$\Phi^{*-1}\{\Phi^*F, \Phi^*G\}_k = \frac{k!}{2^k} p^{m+n-k} J_k^{m,n}(f, g). \quad (12)$$

Proof. The symplectomorphism Φ of \mathbf{R}^2 intertwines the symplectic algebra $sp_2 \equiv sl_2$ and the Möbius algebra: $\Phi^*(p, pq, pq^2) = (\frac{1}{2}p^2, \frac{1}{2}pq, \frac{1}{2}q^2)$. Therefore, the operation $\Phi^{*-1}\{\Phi^*F, \Phi^*G\}_k$ is Möbius-equivariant.

On the other hand, one has: $\Phi^*F = \frac{1}{2^m} p^{2m} f(\frac{q}{p})$ and $\Phi^*G = \frac{1}{2^n} p^{2n} g(\frac{q}{p})$. Since Φ^*F and Φ^*G are homogeneous of degree $2m$ and $2n$ (respectively), the function $\{\Phi^*F, \Phi^*G\}_k$ is also homogeneous of degree $2(m+n-k)$. Thus, the operation $\{F, G\}_k^\Phi = \Phi^{*-1}\{\Phi^*F, \Phi^*G\}_k$ defines a bilinear mapping on the space of tensor-densities $\mathcal{F}_m \otimes \mathcal{F}_n \rightarrow \mathcal{F}_{m+n-k}$ which is sl_2 -equivariant.

Statement 4.1 implies that it is proportional to $J_k^{m,n}$. One easily verifies the coefficient of proportionality for $F = p^m, G = p^n q^k$, to obtain the formula (12).

Proposition 4.3 is proven.

Remark. Proposition 4.3 was proven in [15]. We do not know whether this elementary fact has been mentioned by classics.

4.5 Proof of Theorem 2. Proposition 4.3 implies that the formula (5) is a Φ -conjugation of the Moyal product and is given by the formula (8).

Proposition 1 is proven.

It follows that (5) is a \star -product on \mathcal{F} equivariant under the action of the Möbius sl_2 algebra. Moreover, it is the unique \star -product with this property since the Moyal product is the unique \star -product equivariant under the action of the symplectic algebra.

Theorem 2 is proven.

5 Relation with extensions of the Lie algebra $\text{Vect}(S^1)$

We prove here that the \star -product (5) is not equivalent to the Moyal product.

Let $\text{Vect}(S^1)$ be the Lie algebra of vector fields on the circle. Consider the embedding $\text{Vect}(S^1) \subset \mathcal{F}$ given by functions on \mathbf{R}^2 of the type: $X = pX(q)$ where $X(q)$ is periodical: $X(q+1) = X(q)$.

5.1 An idea of the proof of Theorem 1. Consider the formal deformations of the *Lie algebra* \mathcal{F} associated to the \star -products (1) and (5). The restriction of the Moyal bracket to $\text{Vect}(S^1)$ is identically zero. We show that the restriction of the \star -commutator

$$\{\widetilde{F}, \widetilde{G}\}_t = \frac{1}{i\hbar}(F\widetilde{\star}_\hbar G - G\widetilde{\star}_\hbar F), \quad t = -\frac{\hbar^2}{2}$$

associated to the \star -product (5) defines a series of non-trivial extensions of the Lie algebra $\text{Vect}(S^1)$ by the modules $\mathcal{F}_k(S^1)$ of tensor-densities on S^1 of degree $-k$.

5.2 Extensions and the cohomology group $H^2(\text{Vect}(S^1); \mathcal{F}_\lambda)$. Recall that an *extension* of a Lie algebra by its module is defined by a 2-cocycle on it with values in this module. To define an extension of $\text{Vect}(S^1)$ by the module \mathcal{F}_λ one needs therefore, a bilinear mapping $c : \text{Vect}(S^1)^{\otimes 2} \rightarrow \mathcal{F}_\lambda$ which verify the identity $\delta c = 0$:

$$c(X, [Y, Z]) + L_X^{(\lambda)} c(Y, Z) + (\text{cycle}_{X,Y,Z}) = 0.$$

(see [10]).

The cohomology group $H^2(\text{Vect}(S^1); \mathcal{F}_\lambda)$ were calculated in [19] (see [10]). This group is trivial for each value of λ except $\lambda = 0, -1, -2, -5, -7$. The explicit formulæ for the corresponding non-trivial cocycles are given in [17]. If $\lambda = -5, -7$, then $\dim H^2(\text{Vect}(S^1); \mathcal{F}_\lambda) = 1$, the cohomology group is

generated by the unique (up to equivalence) non-trivial cocycle. We will obtain these cocycles from the \star -commutator.

5.3 Non-trivial cocycles on $\text{Vect}(S^1)$.

Consider the restriction of the \star -commutator $\{\widetilde{}, \widetilde{}\}_t$ (corresponding to the \star -product (5)) to $\text{Vect}(S^1) \subset \mathcal{F}$: let

$$X = pX(q), \quad Y = pY(q),$$

then from (5):

$$\{X, Y\}_t = \{X, Y\} + \sum_{k=1}^{\infty} \frac{t^k}{2^{2k+1}} \frac{1}{p^{2k-1}} J_{2k+1}^{1,1}(X, Y)$$

It follows from the Jacobi identity that the first non-zero term of the series $\{\widetilde{X}, \widetilde{Y}\}_t$ is a 2-cocycle on $\text{Vect}(S^1)$ with values in one of the $\text{Vect}(S^1)$ -modules $\mathcal{F}_k(S^1)$.

Denote for simplicity $J_{2k+1}^{1,1}$ by J_{2k+1} .

From the general formula (4) one obtains:

Lemma 5.1. *First two terms of $\{\widetilde{X}, \widetilde{Y}\}_t$ are identically zero: $J_3(X, Y) = 0$, $J_5(X, Y) = 0$, the next two terms are proportional to:*

$$\begin{aligned} J_7(X, Y) &= X'''Y^{(IV)} - X^{(IV)}Y''' \\ J_9(X, Y) &= 2(X'''Y^{(VI)} - X^{(VI)}Y''') - 9(X^{(IV)}Y^{(V)} - X^{(V)}Y^{(IV)}) \end{aligned} \quad (13)$$

The transvectant J_7 defines therefore a 2-cocycle. It is a remarkable fact that the same fact is true for J_9 :

Lemma 5.2. (see [17]). *The mappings*

$$J_7 : \text{Vect}(S^1)^{\otimes 2} \rightarrow \mathcal{F}_{-5} \quad \text{and} \quad J_9 : \text{Vect}(S^1)^{\otimes 2} \rightarrow \mathcal{F}_{-7}$$

are 2-cocycles on $\text{Vect}(S^1)$ representing the unique non-trivial classes of the cohomology groups $H^2(\text{Vect}(S^1); \mathcal{F}_{-5})$ and $H^2(\text{Vect}(S^1); \mathcal{F}_{-7})$ (respectively).

Proof. Let us prove that J_9 is a 2-cocycle on $\text{Vect}(S^1)$. The Jacobi identity for the bracket $\{\widetilde{}, \widetilde{}\}_t$ implies:

$$\{X, J_9(Y, Z)\} + J_9(X, \{Y, Z\}) + J_3(X, J_7(Y, Z)) + (\text{cycle}_{X,Y,Z}) = 0$$

for any $X = pX(q), Y = pY(q), Z = pZ(q)$. One checks that the expression $J_3(X, J_7(Y, Z))$ is proportional to $X'''(Y'''Z^{(IV)} - Y^{(IV)}Z''')$, and so one gets:

$$J_3(X, J_7(Y, Z)) + (\text{cycle}_{X,Y,Z}) = 0.$$

We obtain the following relation:

$$\{X, J_9(Y, Z)\} + J_9(X, \{Y, Z\}) + (cycle_{X,Y,Z}) = 0$$

which means that J_9 is a 2-cocycle. Indeed, recall that for any tensor density a , $\{pX, p^m a\} = p^m L_{Xd/dx}^{(m)}(a)$. Thus, the last relation coincides with the relation $\delta J_9 = 0$.

Let us show now that the cocycle J_7 on $\text{Vect}(S^1)$ is not trivial. Consider a linear differential operator $A : \text{Vect}(S^1) \rightarrow \mathcal{F}_5$. It is given by: $A(X(q)d/dq) = (\sum_{i=0}^K a_i X^{(i)}(q))(dq)^5$, then $\delta A(X, Y) = L_X^{(5)} A(Y) - L_Y^{(5)} A(X) - A([X, Y])$. The higher order part of this expression has a non-zero term $(5-K)a_K X'Y^{(K)}$ and therefore $J_7 \neq \delta A$.

In the same way one proves that the cocycle J_9 on $\text{Vect}(S^1)$ is non-trivial. Lemma 5.2 is proven.

It follows that the \star -product (5) on the algebra \mathcal{F} is not equivalent to the Moyal product.

Theorem 1 is proven.

6 Operator representation

We are looking for an linear mapping (depending on \hbar) $F \mapsto \widehat{F}^\Phi$ of the associative algebra of Laurent polynomials $\mathcal{F} = C[p, 1/p, q]$ into the algebra of formal pseudodifferential operators on \mathbf{R} such that

$$\widehat{F \star_\hbar G}^\Phi = \widehat{F}^\Phi \widehat{G}^\Phi.$$

Recall that the algebra of Laurent polynomials $C[p, 1/p, q]$ with the Moyal product is isomorphic to the associative algebra of pseudodifferential operators on \mathbf{R} with polynomial coefficients (see [1]). This isomorphism is defined on the generators $p \mapsto \widehat{p}, q \mapsto \widehat{q}$ by the operators (9) and $p^{-1} \mapsto \widehat{p}^{-1}$:

$$\widehat{p}^{-1} = \frac{1}{i\hbar} (\partial/\partial q)^{-1}.$$

6.1 Definition. Put:

$$\widehat{F}^\Phi = \widehat{\Phi^* F} \tag{14}$$

then $\widehat{F}^\Phi \widehat{G}^\Phi = \widehat{\Phi^* F \star_\hbar \Phi^* G} = \widehat{\Phi^* F \star_\hbar \Phi^* G} = \widehat{\Phi^* (F \star_\hbar G)} = \widehat{F \star_\hbar G}^\Phi$.

One obtains the formulæ (10). Indeed,

$$\widehat{p}^\Phi = \widehat{p^2/2} = \frac{(i\hbar)^2}{2} \frac{\partial^2}{\partial q^2}.$$

Since $q = \frac{1}{2} \left(\left(\frac{1}{p} \right) \tilde{\star}_\hbar p q + p q \tilde{\star}_\hbar \left(\frac{1}{p} \right) \right)$, one gets:

$$\hat{q}^\Phi = \frac{1}{4i\hbar} (\Delta \circ A + A \circ \Delta)$$

6.2 sl_2 -equivariance. For the Möbius sl_2 algebra one has:

$$\begin{aligned} \widehat{p}^\Phi &= \frac{(i\hbar)^2}{2} \frac{\partial^2}{\partial q^2} \\ \widehat{pq}^\Phi &= \frac{i\hbar}{4} + \frac{i\hbar}{2} q \frac{\partial}{\partial q} \\ \widehat{pq^2}^\Phi &= \frac{q^2}{2} \end{aligned}$$

Lemma 6.1. *The mapping $F \mapsto \hat{F}^\Phi$ satisfies the Möbius-equivariance condition:*

$$\{\widehat{X}, \widehat{F}\}^\Phi = [\widehat{X}^\Phi, \widehat{F}^\Phi]$$

for $X \in sl_2$.

Proof. It follows immediately from Theorem 2. Indeed, the \star -product (5) is sl_2 -equivariant (that is, satisfying the relation: $\{X, F\}_t = \{X, F\}$ for $X \in sl_2$).

Remark. Beautiful explicit formulæ for sl_2 -equivariant mappings from the space of tensor-densities to the space of pseudodifferential operators are given in [3].

7 Hamiltonian isotopy

The simple calculations below are quite standard for the cohomological technique. We need them to prove Corollary 1 (from Sec. 3).

Given a symplectomorphism Ψ of a symplectic manifold V and a formal deformation $\{, \}_t$ of the Poisson bracket on V , we prove that if Ψ is isotopic to the identity then the formal deformation $\{, \}_t^\Psi$ defined by:

$$\{F, G\}_t^\Psi = \Psi^{*-1} \{ \Psi^* F, \Psi^* G \}_t$$

is equivalent to $\{, \}_t$. The similar proof is valid in the case of \star -products.

Recall that two symplectomorphisms Ψ and Ψ' of a symplectic manifold V are *isotopic* if there exists a family of functions $H_{(s)}$ on V such that the symplectomorphism $\Psi_1 \circ \Psi_2^{-1}$ is the flow of the Hamiltonian vector field with the Hamiltonian function $H_{(s)}$, $0 \leq s \leq 1$.

Let $\Psi_{(s)}$ be the the flow of a family of functions $H = H_{(s)}$. We will prove that the equivalence class of the formal deformation $\{ , \}_t^{\Psi_{(s)}}$ does not depend on s .

7.1 Equivalence of homotopic cocycles. Let us first show that the cohomology class of the cocycle $C_3^{\Psi_{(s)}}$:

$$C_3^{\Psi_{(s)}}(F, G) = \Psi_{(s)}^{*-1} C_3(\Psi_{(s)}^* F, \Psi_{(s)}^* G)$$

does not depend on s . To do this, it is sufficient to prove that the derivative $\dot{C}_3 = \frac{d}{ds} C_3^{\Psi_{(s)}}|_{s=0}$ is a coboundary. One has

$$\dot{C}_3(F, G) = C_3(\{H, F\}, G) + C_3(F, \{H, G\}) - \{H, C_3(F, G)\}.$$

The relation $\delta C_3 = 0$ implies:

$$\dot{C}_3(F, G) = \{F, C_3(G, H)\} - \{G, C_3(F, H)\} - C_3(\{F, G\}, H)$$

This means, $\frac{d}{ds} C_3^{\Psi_{(s)}}|_{s=0} = \delta B_H$, where $B_H(F) = C_3(F, H)$.

7.2 General case. Let us apply the same arguments to prove that the deformations $\{ , \}_t^{\Psi_{(s)}}$ are equivalent to each other for all values of s . We must show that there exists a family of mappings $A_{(s)}(F) = F + \sum_{k=1}^{\infty} A_{(s)_k}(F)t^k$ such that $A_{(s)}^{-1}(\{A_{(s)}(F), A_{(s)}(G)\}_t) = \{F, G\}_t$.

It is sufficient to verify the existence of a mapping $a(F) = \sum_{k=1}^{\infty} a_k(F)t^k$ (the derivative: $a(F) = d/ds(A_{(s)}(F))|_{s=s_0}$) such that

$$\frac{d}{ds} \{F, G\}_t^{\Psi_{(s)}}|_{s=s_0} = \{a(F), G\}_t + \{F, a(G)\} - a(\{F, G\}_t)$$

One has:

$$\frac{d}{ds} \{F, G\}_t^{\Psi_{(s)}}|_{s=s_0} = \{\{F, H\}, G\}_t + \{F, \{G, H\}\}_t - \{\{F, G\}_t, H\}$$

From the Jacobi identity:

$$\{\{F, H\}_t, G\}_t + \{F, \{G, H\}_t\}_t - \{\{F, G\}_t, H\}_t = 0$$

one obtains that the mapping $a(F)$ can be written in the form:

$$a(F) = \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} C_{2k+1}(F, H_{(s_0)})t^k$$

7.3 Proof of Corollary 2. Consider the \star -product (8) given by $F \star_h^\Phi G$, where $F \star_h G$ is the Moyal product (1), $\Phi : (p, q) \mapsto (p^2/2, q/p)$. It is defined on $\mathbf{R}^2 \setminus \mathbf{R}$.

The \star -product (8) on the algebra $C^\infty(\mathbf{R}^2 \setminus \mathbf{R})$ is equivalent to the Moyal product. Indeed, the symplectomorphism Φ is isotopic to the identity in the group of all smooth symplectomorphisms of $\mathbf{R}^2 \setminus \mathbf{R}$. The isotopy is: $\Phi_s : (p, q) \mapsto (\frac{p^{1+s}}{1+s}, \frac{q}{p^s})$, where $s \in [0, 1]$.

Recall, that the \star -product $F \star_h^\Phi G$ on the algebra \mathcal{F} is not equivalent to the Moyal product since it coincides with the product (5).

The family Φ_s does not preserve the algebra \mathcal{F} . Theorem 1 implies that F is not isotopic to the identity in the group of symplectomorphisms of $\mathbf{R}^2 \setminus \mathbf{R}$ preserving the algebra \mathcal{F} .

8 Discussion

8.1 Difficulties in multi-dimensional case

There exist multi-dimensional analogues of transvectants [14] and [16].

Consider the projective space \mathbf{RP}^{2n+1} endowed with the standard contact structure (or an open domain of the complex projective space \mathbf{CP}^{2n+1}). There exists an unique bilinear differential operator of order k on tensor-densities equivariant with respect to the action of the group Sp_{2n} (see [14], [16]):

$$J_k^{\lambda, \mu} : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_{\lambda+\mu-\frac{k}{n+1}} \quad (15)$$

where $\mathcal{F}_\lambda = \mathcal{F}_\lambda(\mathbf{P}^{2n+1})$ is the space of tensor-densities on \mathbf{P}^{2n+1} of degree $-\lambda$:

$$f = f(x_1, \dots, x_{2n+1})(dx_1 \wedge \dots \wedge dx_{2n+1})^{-\lambda}.$$

The space of tensor-densities $\mathcal{F}_\lambda(\mathbf{RP}^{2n+1})$ is *isomorphic as a module over the group of contact diffeomorphisms* to the space of homogeneous functions on \mathbf{R}^{2n+2} . The isomorphism is given by:

$$f \mapsto F(y_1, \dots, y_{2n+2}) = y_{2n+2}^{-\lambda(n+1)} f\left(\frac{y_1}{y_{2n+2}}, \dots, \frac{y_{2n+1}}{y_{2n+2}}\right)$$

Then, the operations (15) are defined as the restrictions of the terms of the standard \star -product on \mathbf{R}^{2n+2} .

The same formula (5) defines a \star -product on the space of tensor-densities on \mathbf{CP}^{2n+1} (cf. [16]). However, there is no analogues of the symplectomorphism (7). I do not know if there exists a \star -product on the Poisson algebra $C[y_{2n+2}, y_{2n+2}^{-1}] \otimes C^\infty(\mathbf{RP}^{2n+1})$ non-equivalent to the standard.

8.2 Classification problem. The classification (modulo equivalence) of \star -products on the Poisson algebra \mathcal{F} is an interesting open problem. It is related to calculation of cohomology groups $H^2(\mathcal{F}; \mathcal{F})$ and $H^3(\mathcal{F}; \mathcal{F})$. The following result is announced in [7]: $\dim H^2(\mathcal{F}; \mathcal{F}) = 2$.

Let us formulate a conjecture in the compact case. Consider the Poisson algebra $\mathcal{F}(S^1)$ of functions on $T^*S^1 \setminus S^1$ which are Laurent polynomials on the fiber: $F(p, q) = \sum_{-N \leq i \leq N} p^i f_i(q)$ where $f_i(q+1) = f_i(q)$.

Conjecture. *Every \star -product on $\mathcal{F}(S^1)$ is equivalent to (1) or to (5).*

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